March 23: Nagata Rings, part 1

Discussion

In this part of the course we deal with the following question: Given a Noetherian domain R with quotient field K, when is the integral closure of R in a finite extension of K a finite R-module?

The rings from algebraic geometry have this property.

Our immediate goal is to see to what extent this property holds in a purely algebraic setting.

Until further notice, or unless indicated otherwise, R will denote a Noetherian integral domain with quotient field K.

We will consistently use L to denote a finite field extension of K and S to denote the integral closure of R in L.

For an integral domain T, we will write T' for the integral closure of T (in its quotient field).

Definition

Maintaining the notation above:

1. R is said to satisfy N_1 if R' is a finite R-module.

2. R is said to satisfy N_2 if S' is a finite S-module for all finite extensions L of K.

3. R is said to be a *Nagata ring* if R/P satisfies N_2, for all prime ideals $P \subseteq R$.

Comments

1. The main goal of this section is to prove that if R is a Nagata ring, then any finitely generated R-algebra T is a Nagata ring.

2. Though we are assuming throughout that R is an integral domain, the definition of Nagata ring clearly applies to any Noetherian ring.

The theorem we seek for arbitrary rings reduces trivially to integral domains, so we do not lose any generality by assuming R and T are integral domains.

Observations

For R as above:

1. If R satisfies N_2, then any finite integral extension of R satisfies N_2, and conversely.

2. If R satisfies N_2 or is a Nagata ring, then R_S is N_2 or a Nagata ring, for any multiplicatively closed set $S \subseteq R$.

3. If R is a Nagata ring, any finite extension is a Nagata ring, and conversely. After all:

For any such extension S and prime $Q \subseteq S$, $R/(Q \cap R) \subseteq S/Q$ is finite.

4. If R is a complete local domain, R is a Nagata ring. Why: Each factor R/P is a complete local domain and complete local domains satisfy N_2 (coming soon).

Integral closure and principal ideals

We will require a number of preliminary results before getting to the main result on Nagata rings. For this, we need a good understanding of integral closure. Our first result is a variation on Serre's criteria for a ring to be integrally closed.

Proposition A. Let $0 \neq x \in R$. Then xR is integrally closed if and only if R_P is a DVR, for all $P \in \mathrm{Ass}(R/xR)$.

Proof. Suppose that for each $P \in \mathrm{Ass}(R/xR)$, R_P is a DVR, and fix one such P. If $y \in \overline{xR}$, then $y \in \overline{xR_P} = xR_P$, since R_P is a DVR. Since this holds for all P, $y \in xR$.

Now suppose xR is integrally closed and $P \in \mathrm{Ass}(R/xR)$. We may assume R is local at P. Write $P = (xR : a)$. Note: $P = (R : \frac{a}{x})$.

$P \cdot \frac{a}{x}$ is an ideal of R. If $P \cdot \frac{a}{x} \subseteq P$, then by the *determinant trick*, $\frac{a}{x}$ is integral over R.

Integral closure and principal ideals, continued

Thus, $a \in \overline{xR} = xR$, a contradiction. Therefore $P \cdot \frac{a}{x} = R$.
Take $p_0 \in P$ such that $p_0 \cdot \frac{a}{x} = 1$.

Now take any $p \in P$. Note that $p \cdot \frac{a}{x} \in R$.

Thus, $p = (p \cdot \frac{a}{x}) \cdot p_0$, which shows $P = p_0 R$.

Therefore, R is a DVR.

Corollary B. For R as above, R is integrally closed if and only if for every prime $P \subseteq R$ associated to a principal ideal, R_P is a DVR.

Proof. Since an element $\frac{a}{x} \in K$ is integral over R if and only if $a \in \overline{xR}$, R is integrally closed if and only if each principal ideal xR is integrally closed.

Thus, the corollary follows immediately from Theorem A.

Integral closure and principal ideals, continued

Corollary C. Suppose there exists $0 \neq x \in R$ such that R_x is integrally closed. Then there exists and ideal $J \subseteq R$ such that for all prime ideals $Q \subseteq R$, R_Q is integrally closed if and only if $J \not\subseteq Q$.

Proof. If R_P is a DVR for all $P \in \mathrm{Ass}(R/xR)$, then the previous corollary implies that R is integrally closed and we just take $J = R$.

Otherwise, let $P_1, \ldots, P_r$ be the prime ideals in $\mathrm{Ass}(R/xR)$ such that R_P is NOT a DVR. Set $J := P_1 \cap \cdots \cap P_r$.

Suppose $Q \subseteq R$ is a prime ideal. If $J \subseteq Q$, then $P_i \subseteq Q$, some i. Since R_Q localized at P_iR_Q is just R_{P_i}, xR_Q is not integrally closed, and thus R_Q is not integrally closed.

Suppose R_Q is not integrally closed. Then we must have $x \in Q$ and xR_Q is not integrally closed. By standard localization arguments, $P_i \subseteq Q$, for some i.

Thus, $J \subseteq Q$.

Integral closure and principal ideals, continued

In general, if T is a Noetherian domain and $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$, then it need not be the case that T' is finite over T. However, the next important result gives a case when this does hold.

Theorem D. Let T be a Noetherian domain satisfying the properties:

(a) T_b is integrally closed for some $0 \neq b \in T$.

(b) $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ containing b.

Then T' is finite over T.

Remark. Note that conditions (a) and (b) above together imply that $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.

Proof. For each maximal ideal $\mathfrak{m}$ containing b, let $T \subseteq T(\mathfrak{m}) \subseteq T'$ be a ring which is a finite T-module satisfying $T(\mathfrak{m})_{\mathfrak{m}} = T'_{\mathfrak{m}}$.

Fix $\mathfrak{m}$.

Integral closure and principal ideals, continued

Since $T_b = T'_b$, $T(\mathfrak{m})_b = T'_b$ is integrally closed.

By the previous corollary, there exists $J(\mathfrak{m}) \subseteq T(\mathfrak{m})$ such that for all primes $Q \subseteq T(\mathfrak{m})$, T_Q is integrally closed if and only if $J(\mathfrak{m}) \not\subseteq Q$.

Set $I(\mathfrak{m}) = J(\mathfrak{m}) \cap T$. Note that since $T(\mathfrak{m})_{\mathfrak{m}}$ is integrally closed, $I(\mathfrak{m}) \not\subseteq \mathfrak{m}$, since $J(\mathfrak{m})_{\mathfrak{m}} = T(\mathfrak{m})_{\mathfrak{m}}$.

Let $J \subseteq T$ be the ideal such that T_Q is integrally closed if and only if $J \not\subseteq Q$. Thus $J \not\subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ not containing b.

If we take the sum of all ideals $I(\mathfrak{m})$ together with J, then we get an ideal not contained in any maximal ideal of T. Thus, this sum equals T.

Thus a finite set of ideals from this collection sum to T. Call these ideals $J, I(\mathfrak{m}_1), \ldots, I(\mathfrak{m}_s)$.

Note that it does no harm to include J, even if it is not required.

Set $\widetilde{T} := T[T(\mathfrak{m}_1), \cdots, T(\mathfrak{m}_s)]$, a finite T-module with $T \subseteq \widetilde{T} \subseteq T'$.

Integral closure and principal ideals, continued

We claim $\widetilde{T} = T'$. It suffices to show that $\widetilde{T}_Q = T'_Q$ for all maximal ideals $Q \subseteq \widetilde{T}$. Fix a maximal ideal Q.

Set $\mathfrak{m} := Q \cap T$. Then $\mathfrak{m}$ does not contain $J + I(\mathfrak{m}_1) + \cdots + I(\mathfrak{m}_s)$.

If $J \not\subseteq \mathfrak{m}$, then $T_\mathfrak{m} = T'_\mathfrak{m}$. Hence $\widetilde{T}_\mathfrak{m} = T'_\mathfrak{m}$, and therefore $\widetilde{T}_Q = T'_Q$.

If $I(\mathfrak{m}_i) \not\subseteq \mathfrak{m}$, set $Q_0 := Q \cap T(\mathfrak{m}_i)$. Then $J(\mathfrak{m}_i) \not\subseteq Q_0$.

Thus, $T(\mathfrak{m}_i)_{Q_0} = T'_{Q_0}$. Therefore $\widetilde{T}_{Q_0} = T'_{Q_0}$.

Since $\widetilde{T}_Q$ and T'_Q are further localizations of $\widetilde{T}_{Q_0} = T'_{Q_0}$, it follows that $\widetilde{T}_Q = T'_Q$, as required. □

Integral closure and principal ideals, continued

Corollary E. Suppose that R is integrally closed and locally analytically unramified. If $R \subseteq T \subseteq K$ is a finitely generated R-algebra, then T' is a finite T-module.

Proof. We can write $T = R[\frac{a_1}{b}, \ldots, \frac{a_n}{b}]$, for $a_i, b \in R$. Then, $T_b = R_b$ is integrally closed.

On the other hand, let $\mathfrak{m} \subseteq T$ be a maximal ideal and set $Q := \mathfrak{m} \cap R$. Then T'_Q is finite over T_Q, since R_Q is analytically unramified.Thus, $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$, so the result follows from Theorem D.

Integral closure in characteristic zero

Our next result shows that there is no difference between the conditions N_1 and N_2 for rings having characteristic zero.

Theorem F. Suppose that R has characteristic zero and satisfies condition N_1. Then R satisfies N_2. In particular, if R is integrally closed, and has characteristic zero, then R satisfies N_2.

Proof. Let L be a finite extension of K and S the integral closure of R in L.

If we show that S is a finite R'-module, then since R satisfies N_1, S is a finite R-module.

Thus, it suffices to prove the second statement.

In fact: At this point we do not need to assume R has characteristic zero, only that the extension is separable.